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GENERALIZED CHARACTERISTICS AND THE STRUCTURE OF SOLUTIONS OF H--ETC(U)
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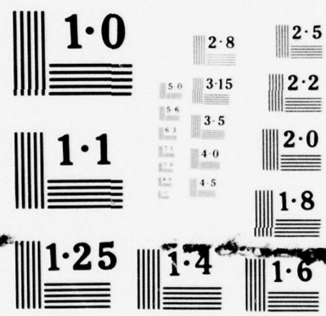

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GENERALIZED CHARACTERISTICS AND THE STRUCTURE OF SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS+

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by

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15 N00014-76-C-0278, NSF-MPS-71-02923

+This research was supported in part by the U. S. Army under AROD AAG 29-76-6-0052, in part by the Office of Naval Research under NONR N000 14-76-C-0278 and in part by the National Science Foundation under MPS-71-02923.

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1. Introduction

Consider the conservation law

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t),x,t)}{\partial x} + g(u(x,t),x,t) = 0 \qquad (1.1)$$

where $f(u,x,t)$ and $g(u,x,t)$ are, respectively, C^2 and C^1 smooth functions and, for fixed (x,t), $f(u,x,t)$ is strictly (but not necessarily uniformly) convex in u, i.e., $f_{uu}(u,x,t)$ is non-negative and does not vanish identically on any u-interval.

In general, the initial-value problem for (1.1) does not have global smooth solutions even if the initial data are C^∞ smooth. Classical prototypes of weak solutions are constructed in the class of piecewise smooth functions with jump discontinuities across smooth curves. In Continuum Physics (that has motivated the study of conservation laws) solutions in this class have a natural interpretation, the lines of jump discontinuity being inerpreted as trajectories of propagating shock waves. Unfortunately, the class of piecewise smooth functions is too narrow to encompass all solutions of (1.1), even those with C^∞ smooth initial data.

Research on hyperbolic conservation laws has revealed that the natural framework in which solutions should be studied is provided by the class of functions of bounded variation in the sense of Tonelli and Cesari(*). Volpert [3] has called attention

(*) The relevance of this class was first recognized by Conway and Smoller [1]. The importance of estimates involving variation in the study of (1.1) had been demonstrated earlier by Oleinik (e.g. [2]).

to the analogy between the geometric structure of piecewise smooth functions and that of functions of bounded variation, with points of smoothness and jump discontinuity in the former class corresponding to points of approximate continuity and approximate jump discontinuity in the latter. This structure is inherited by solutions of hyperbolic conservation laws. The natural question, of course, is whether membership in the class of functions of bounded variation provides maximal information on the structure of solutions. The available results indicate that this is not the case.

In the special case $g \equiv 0$, there is a connection between (1.1) and the Hamilton-Jacobi equation which induces an explicit representation of solutions. Using this representation, Oleinik [4,5,2] shows that solutions of (1.1) with $g \equiv 0$ are continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks). Analogous results were subsequently established by DiPerna [6] for solutions of homogeneous, genuinely nonlinear, systems of hyperbolic conservation laws constructed by Glimm's scheme [7,8].

The above results do not preclude the possibility of solutions with quite complicated structure, e.g. with shock sets that are everywhere dense on the domain of definition. In the homogeneous case

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0 \qquad (1.2)$$

and by using Thom's theory of catastrophes in conjunction with an explicit construction of solutions due to Lax [9], Schaeffer [10]

shows that generically solutions with C^∞ smooth initial data are piecewise smooth (see also the related papers [11,12]).

In contrast to the approaches cited above which are associated with specific construction schemes for solutions, our point of view here is to study solutions of (1.1) a priori. Namely, we consider a solution in an appropriate function class introduced in Section 2 (somewhat broader than the class of functions of bounded variation), with no reference to how this solution was constructed, and we proceed to study its structure.

Pursued for systems, such a program could provide the key to the solution of the open problem of uniqueness of solutions. However, although many of the arguments used here can be extended routinely to systems, one key result (Theorem 3.3) seems to be valid only for the single equation so that new ideas will be needed in order to carry out the extension.

The main tool in implementing our program is the concept of a generalized characteristic. A characteristic of (1.1), associated with a classical solution $u(x,t)$, is defined as a trajectory of the ordinary differential equation

$$\frac{dx}{dt} = f_u(u(x,t),x,t). \tag{1.3}$$

For weak solutions the same definition is adopted here where, however, (1.3) is now interpreted as a contingent equation in the sense of Filippov [13]. In Section 3, we show that generalized characteristics are either classical characteristics or shocks. From each point (x,t) there emanates a unique forward characteristic. On the other hand, the set of backward characteristics

through (x,t) either consists of a single, classical characteristic or it contains an infinite number of curves spanning the funnel confined between two classical characteristics.

Using the above information we proceed in Section 4 to the study of the structure of solutions. We prove that every shock is a Lipschitz continuous curve which is differentiable except at points of interaction with other shocks or centered compression waves. The set of shocks is at most countable and the solution is continuous on the complement of the shock set.

In Section 5, we study the smoothness of solutions with smooth initial data. We show that the complement of the shock set is open and the solution is smooth on this set. We are also able to characterize, by elementary methods, the set of smooth initial data that may generate solutions which are not piecewise smooth and to show that this set is of the first category. Thus, solutions of (1.1) with smooth initial data are generically piecewise smooth.

The concept of generalized characteristic is also useful in studying other properties of solutions of hyperbolic conservation laws. For example, in [14] we use the same methodology to discuss the asymptotic behavior of solutions of (1.2).

Acknowledgement. I am indebted to Ronald DiPerna for many stimulating discussions and fruitful suggestions.

2. Admissible Solutions

We introduce here the class of functions in which admissible solutions of (1.1) will be considered.

Definition 2.1. A measurable, locally bounded function $u(x,t)$ on $(-\infty,\infty) \times [0,\infty)$ is of class $\mathscr{K}$ if $t \to u(\cdot,t)$, as a map from $[0,\infty)$ to $L^1_{loc}(-\infty,\infty)$, is weakly continuous and for almost all $t \in [0,\infty)$ one-sided limits $u(x-,t)$, $u(x+,t)$ exist for all $x \in (-\infty,\infty)$.

Remark 2.1. Functions of locally bounded variation in the sense of Tonelli and Cesari [3] are of class $\mathscr{K}$. In particular, if $u(x,t)$ is of locally bounded variation, then $u(\cdot,t)$ is of locally bounded variation on $(-\infty,\infty)$ for almost all (fixed) $t \in [0,\infty)$.

Definition 2.2. A function $u(x,t)$ of class $\mathscr{K}$ which satisfies (1.1) in the sense of distributions is an admissible solution if for almost all $t \in [0,\infty)$,

$$u(x-,t) \geq u(x+,t) \tag{2.1}$$

for all $x \in (-\infty,\infty)$.

The interpretation of the admissibility condition (2.1) (sometimes called "entropy condition") is of course familiar.

3. Characteristics

Throughout this section $u(x,t)$ will denote an admissible solution of (1.1).

Definition 3.1. A Lipschitz continuous curve $\xi(\cdot): [a,b] \to (-\infty,\infty)$ is called a characteristic if for almost all $t \in [a,b]$

$$\dot{\xi}(t) \in [f_u(u(\xi(t)+,t),\xi(t),t), f_u(u(\xi(t)-,t),\xi(t),t)]. \qquad (3.1)$$

By the theory of contingent equations of type (3.1) [13], through any fixed point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ there is at least one forward characteristic, defined on $[\bar{t},\infty)$, and at least one backward characteristic, defined on $[0,\bar{t}]$. The set of forward (or backward) characteristics through $(\bar{x},\bar{t})$ spans the funnel confined between a minimal and a maximal forward (or backward) characteristic through $(\bar{x},\bar{t})$. Of course, the minimal and maximal forward (or backward) characteristics are not necessarily distinct. The minimal and maximal backward characteristics through $(\bar{x},\bar{t})$ will play a central role in this paper and will be denoted throughout by $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$, respectively.

Seemingly (3.1) allows considerable freedom in the speed of propagation of characteristics. The following propostion shows, however, that characteristics must propagate either at classical characteristic speed or at shock speed.

<u>Theorem 3.1</u>. Let $\xi(\cdot): [a,b] \to (-\infty,\infty)$ be a characteristic. Then for almost all $t \in [a,b]$

$$\dot{\xi}(t) = \begin{cases} f_u(u(\xi(t)\pm,t),\xi(t),t) & \text{if } u(\xi(t)-,t) = u(\xi(t)+,t) \\ \dfrac{f(u(\xi(t)+,t),\xi(t),t)-f(u(\xi(t)-,t),\xi(t),t)}{u(\xi(t)+,t)-u(\xi(t)-,t)} & \text{if } u(\xi(t)-,t) > u(\xi(t)+,t). \end{cases} \tag{3.2}$$

Theorem 3.1 is an immediate corollary of (3.1) and Lemma 3.1, below.

<u>Lemma 3.1</u>. Let $\xi(\cdot): [a,b] \to (-\infty,\infty)$, $0 \leq a < b < \infty$, be a Lipschitz continuous curve. Then, for almost all $t \in [a,b]$,

$$\begin{aligned} f(u(\xi(t)+,t),\xi(t),t) &- f(u(\xi(t)-,t),\xi(t),t) \\ &- \dot{\xi}(t)[u(\xi(t)+,t) - u(\xi(t)-,t)] = 0. \end{aligned} \tag{3.3}$$

<u>Lemma 3.2</u>. Let $\xi(\cdot): [a,b] \to (-\infty,\infty)$ and $\zeta(\cdot): [a,b] \to (-\infty,\infty)$, $0 \leq a < b < \infty$, be Lipschitz continuous curves. Then, for any σ,τ, $a \leq \sigma < \tau \leq b$,

$$\begin{aligned} \int_{\xi(\tau)}^{\zeta(\tau)} u(x,\tau)dx - \int_{\xi(\sigma)}^{\zeta(\sigma)} u(x,\sigma)dx + \int_\sigma^\tau \int_{\xi(t)}^{\zeta(t)} g(u(x,t),x,t)dxdt \\ = \int_\sigma^\tau \{f(u(\xi(t)-,t),\xi(t),t) - \dot{\xi}(t)u(\xi(t)-,t)\}dt \\ - \int_\sigma^\tau \{f(u(\zeta(t)+,t),\zeta(t),t) - \dot{\zeta}(t)u(\zeta(t)+,t)\}dt. \end{aligned} \tag{3.4}$$

Proof of Lemmas 3.1 and 3.2. Assume first that $\xi(t) \leq \zeta(t)$, $t \in [a,b]$. Since $u(x,t)$ is a solution of (1.1) in the sense of distributions,

$$\int_0^\infty \int_{-\infty}^\infty [u\phi_t + f\phi_x - g\phi]\,dx\,dt = 0 \tag{3.5}$$

for any Lipschitz continuous function $\phi(x,t)$ defined on $(-\infty,\infty) \times (-\infty,\infty)$ and having compact support contained in $(-\infty,\infty) \times [0,\infty)$. For ε positive small we define

$$\phi_\varepsilon(x,t) = \psi_\varepsilon(x,t)\chi_\varepsilon(t) \tag{3.6}$$

where

$$\chi_\varepsilon(t) = \begin{cases} 0 & t < \sigma \\ \dfrac{t-\sigma}{\varepsilon} & \sigma < t < \sigma + \varepsilon \\ 1 & \sigma + \varepsilon \leq t < \tau \\ 1 + \dfrac{\tau - t}{\varepsilon} & \tau \leq t < \tau + \varepsilon \\ 0 & \tau + \varepsilon \leq t \end{cases} \tag{3.7}$$

$$\psi_\varepsilon(x,t) = \begin{cases} 0 & x < \xi(t) - \varepsilon \\ 1 + \dfrac{x - \xi(t)}{\varepsilon} & \xi(t) - \varepsilon \leq x < \xi(t) \\ 1 & \xi(t) \leq x \leq \zeta(t) \\ 1 + \dfrac{\zeta(t) - x}{\varepsilon} & \zeta(t) < x \leq \zeta(t) + \varepsilon \\ 0 & \zeta(t) + \varepsilon < x. \end{cases} \tag{3.8}$$

Applying (3.5) for the test function $\phi_\varepsilon(x,t)$, letting $\varepsilon \to 0+$ and recalling that $u(x,t)$ is of class $\mathscr{K}$ (Definition 2.1) we arrive at (3.4) for the case $\xi(t) \le \zeta(t)$, $t \in [a,b]$.

In particular, applying (3.4) for $\xi(t) \equiv \zeta(t)$ we obtain (3.3) thus proving Lemma 3.1.

In the general case where $\xi(t) - \zeta(t)$ becomes positive at some (or at all) points of $[a,b]$, we first write (3.4) for the functions $\bar{\xi}(t) \overset{\text{def}}{=} \min\{\xi(t),\zeta(t)\}$, $\bar{\zeta}(t) \overset{\text{def}}{=} \max\{\xi(t),\zeta(t)\}$, $t \in [a,b]$, and then use Lemma 3.1 to derive (3.4) for $\xi(\cdot)$ and $\zeta(\cdot)$. This completes the proof.

<u>Definition 3.2</u>. A characteristic $\xi(\cdot)$ on $[a,b]$ is called <u>genuine</u> if $u(\xi(t)-,t) = u(\xi(t)+,t)$ for almost all $t \in [a,b]$.

The following proposition shows that every point of the upper half-plane can be joined to the axis $t = 0$ by a genuine characteristic.

<u>Theorem 3.2</u>. For any $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$, the minimal and maximal backward characteristics $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$ through $(\bar{x},\bar{t})$ are genuine.

<u>Proof</u>. We abbreviate $\zeta_-(t;\bar{x},\bar{t})$ by $\zeta(t)$ and we prove that it is genuine on $[0,\bar{t}]$. If $\zeta(\cdot)$ is not genuine we can find a measurable set $J \subset [0,\bar{t}]$, with $\mu(J) > 0$ (*), and a number $\beta > 0$ such that $u(\zeta(t)-,t) - u(\zeta(t)+,t) > \beta$ for $t \in J$. Since f is

(*) Here and throughout μ and μ^* denote, respectively, the one-dimensional Lebesgue measure and outer measure.

strictly convex in u, there is $\varepsilon > 0$ such that for $t \in J$

$$f_u(u(\zeta(t)-,t),\zeta(t),t) - \frac{f(u(\zeta(t)+,t),\zeta(t),t)-f(u(\zeta(t)-,t),\zeta(t),t)}{u(\zeta(t)+,t)-u(\zeta(t)-,t)} < 2\varepsilon. \quad (3.9)$$

For each $t \in J$ there exists $\delta(t) > 0$ with the property

$$f_u(u(x+,t),x,t) \geq f_u(u(\zeta(t)-,t),\zeta(t),t) - \varepsilon, \quad \zeta(t) - \delta(t) < x < \zeta(t). \quad (3.10)$$

Finally, there is a subset I of J with $\mu^*(I) > 0$ and $\overline{\delta} > 0$ such that $\delta(t) \geq \overline{\delta}$ for $t \in I$.

Let τ be a density point of I, with respect to μ^*. We can thus find $\overline{r}$, $0 < \overline{r} < \overline{t} - \tau$, so that

$$\frac{\mu^*(I \cap [\tau,\tau+r])}{r} > \frac{2|\alpha|+\varepsilon}{2|\alpha|+2\varepsilon}, \quad 0 < r \leq \overline{r}, \quad (3.11)$$

where

$$\alpha \stackrel{\text{def}}{=} \inf\{f_u(u(x+,t),x,t)-f_u(u(\zeta(t)-,t),\zeta(t),t) \mid 0 \leq t \leq \overline{t}, \\ \zeta(t) - \overline{\delta} \leq x < \zeta(t)\}. \quad (3.12)$$

We now fix a point $y \in (\zeta(\tau) - \overline{\delta}, \zeta(\tau))$ with

$$y > \zeta(\tau) - \frac{\varepsilon\overline{r}}{2} \quad (3.13)$$

and we consider a forward characteristic $\xi(\cdot)$ through (y,τ). We first observe that since, by hypothesis, $\zeta(\cdot)$ is the minimal backward characteristic through $(\overline{x},\overline{t})$,

$$\xi(t) < \zeta(t), \quad t \in [\tau, \bar{t}], \tag{3.14}$$

and we proceed to show that this leads to a contradiction.

We begin by showing that

$$\xi(t) > \zeta(t) - \bar{\delta}, \ t \in [\tau, \tau+\bar{r}]. \tag{3.15}$$

Indeed, suppose that for some $r \in (0,\bar{r}]$, $\xi(t) > \zeta(t) - \bar{\delta}$, $t \in [\tau,\tau+r)$, but $\xi(\tau+r) = \zeta(\tau+r) - \bar{\delta}$. Then, using (3.1), (3.2), (3.9), (3.10), (3.11) and (3.12) we obtain

$$0 = \xi(\tau+r) - \zeta(\tau+r) + \bar{\delta} = y + \int_{\tau}^{\tau+r} \dot{\xi}(t)dt - \zeta(\tau) - \int_{\tau}^{\tau+r} \dot{\zeta}(t)dt + \bar{\delta}$$
$$> \varepsilon\mu^*(I \cap [\tau,\tau+r]) + \alpha[r-\mu^*(I \cap [\tau,\tau+r])] > 0,$$

namely a contradiction that verifies (3.15).

On account of (3.14) and (3.15), using again (3.1), (3.2), (3.9), (3.10), (3.11), (3.12) and (3.13),

$$0 > \xi(\tau+\bar{r}) - \zeta(\tau+\bar{r}) = y + \int_{\tau}^{\tau+\bar{r}} \dot{\xi}(t)dt - \zeta(\tau) - \int_{\tau}^{\tau+\bar{r}} \dot{\zeta}(t)dt$$
$$> \varepsilon\mu^*(I \cap [\tau,\tau+\bar{r}]) + \alpha[\bar{r}-\mu^*(I \cap [\tau,\tau+\bar{r}])] - \frac{\varepsilon\bar{r}}{2} > 0$$

and this contradiction completes the proof of the assertion that $\zeta_-(t;\bar{x},\bar{t})$ is genuine. The proof that $\zeta_+(t;\bar{x},\bar{t})$ is also genuine is similar and will thus be omitted.

The proof of Theorem 3.2 does not depend essentially on that we are dealing here with a single conservation law but only on the

observation that the convexity of f in u together with the admissibility condition (2.1) imply that the classical characteristic speed on the left (or right) side of a shock is greater (or less) than the shock speed [9]. An analogous result will thus hold for general genuinely nonlinear systems of hyperbolic conservation laws. For conservation laws that are not genuinely nonlinear, the minimal and maximal backward characteristics shall be composed of genuine characteristics and/or contact discontinuities.

In contrast, the next proposition which asserts that genuine characteristics are classical characteristics, relies heavily on the special structure of equation (1.1) as well as on the convexity hypothesis on f.

<u>Theorem 3.3</u>. Let $\xi(\cdot)$ be a genuine characteristic on $[a,b]$. Then there is a function $v(\cdot)$ on $[a,b]$ so that $(\xi(\cdot),v(\cdot))$ is a (continuously differentiable) solution of the classical characteristic equations

$$\begin{aligned} \dot{\xi} &= f_u(v,\xi,t) \\ \dot{v} &= -f_x(v,\xi,t) - g(v,\xi,t). \end{aligned} \tag{3.16}$$

Furthermore,

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)-\varepsilon}^{\xi(t)} u(x,t)dx = v(t) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)}^{\xi(t)+\varepsilon} u(x,t)dx, \quad t \in (a,b), \tag{3.17}$$

$$\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(a)-\varepsilon}^{\xi(a)} u(x,a)dx \le v(a) \le \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(a)}^{\xi(a)+\varepsilon} u(x,a)dx, \tag{3.18}$$

$$\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(b)-\varepsilon}^{\xi(b)} u(x,b)dx \geq v(b) \geq \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(b)}^{\xi(b)+\varepsilon} u(x,b)dx. \tag{3.19}$$

In particular,

$$u(\xi(t)-,t) = v(t) = u(\xi(t)+,t), \quad \text{a.e. on } [a,b]. \tag{3.20}$$

<u>Proof</u>. Since $\xi(\cdot)$ is genuine on $[a,b]$, there is a subset J of $[a,b]$, of total measure, such that if $t \in J$, $u(\xi(t)-,t) = u(\xi(t)+,t)$. We thus define a bounded measurable function $v(\cdot)$ on J so that (3.20) is satisfied for all $t \in J$. Furthermore, (3.1) yields

$$\dot{\xi}(t) = f_u(v(t),\xi(t),t), \quad \text{a.e. on } [a,b]. \tag{3.21}$$

We now fix σ,τ, $a \leq \sigma < \tau \leq b$, and $\varepsilon > 0$. Applying (3.4) with $\zeta(t) = \xi(t) - \varepsilon$ we obtain

$$\begin{aligned}
&\int_{\xi(\tau)-\varepsilon}^{\xi(\tau)} u(x,\tau)dx - \int_{\xi(\sigma)-\varepsilon}^{\xi(\sigma)} u(x,\sigma)dx + \int_\sigma^\tau \int_{\xi(t)-\varepsilon}^{\xi(t)} g(u(x,t),x,t)dxdt \\
&= \int_\sigma^\tau \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f_u(v(t),\xi(t),t)u(\xi(t)-\varepsilon+,t)\}dt \\
&- \int_\sigma^\tau \{f(v(t),\xi(t),t) - f_u(v(t),\xi(t),t)v(t)\}dt.
\end{aligned} \tag{3.22}$$

Recalling that f is convex in u, we deduce from (3.22)

$$\begin{aligned}
&\int_{\xi(\tau)-\varepsilon}^{\xi(\tau)} u(x,\tau)dx - \int_{\xi(\sigma)-\varepsilon}^{\xi(\sigma)} u(x,\sigma)dx + \int_\sigma^\tau \int_{\xi(t)-\varepsilon}^{\xi(t)} g(u(x,t),x,t)dxdt \\
&\geq \int_\sigma^\tau \{f(u(\xi(t)-\varepsilon+,t),\xi(t)-\varepsilon,t) - f(u(\xi(t)-\varepsilon+,t),\xi(t),t)\}dt.
\end{aligned} \tag{3.23}$$

Dividing by ε and letting $\varepsilon \to 0+$,

$$\lim_{\varepsilon\to 0+} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{\varepsilon}\int_{\xi(\tau)-\varepsilon}^{\xi(\tau)} u(x,\tau)dx \geq \lim_{\varepsilon\to 0+} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{\varepsilon}\int_{\xi(\sigma)-\varepsilon}^{\xi(\sigma)} u(x,\sigma)dx$$

$$- \int_{\sigma}^{\tau} g(v(t),\xi(t),t)dt - \int_{\sigma}^{\tau} f_x(v(t),\xi(t),t)dt. \tag{3.24}$$

Similarly, applying (3.4) with $\zeta(t) = \xi(t) + \varepsilon$ and following the above procedure we arrive at

$$\lim_{\varepsilon\to 0+} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{\varepsilon}\int_{\xi(\tau)}^{\xi(\tau)+\varepsilon} u(x,\tau)dx \leq \lim_{\varepsilon\to 0+} \begin{matrix}\sup\\ \inf\end{matrix} \frac{1}{\varepsilon}\int_{\xi(\sigma)}^{\xi(\sigma)+\varepsilon} u(x,\sigma)dx$$

$$- \int_{\sigma}^{\tau} g(v(t),\xi(t),t)dt - \int_{\sigma}^{\tau} f_x(v(t),\xi(t),t)dt. \tag{3.25}$$

Applying (3.24), (3.25) for $\sigma, \tau \in J$, the set on which (3.20) is satisfied,

$$v(\tau) = v(\sigma) - \int_{\sigma}^{\tau} g(v(t),\xi(t),t)dt - \int_{\sigma}^{\tau} f_x(v(t),\xi(t),t)dt \tag{3.26}$$

which shows that $v(\cdot)$ can be extended on $[a,b]$ as a continuously differentiable function satisfying the differential equation

$$\dot{v}(t) = -f_x(v(t),\xi(t),t) - g(v(t),\xi(t),t), \quad t \in [a,b]. \tag{3.27}$$

Then (3.21) implies that $\xi(\cdot)$ is also continuously differentiable on $[a,b]$ and $(\xi(\cdot),v(\cdot))$ is a solution of (3.16).

We now fix $t \in [a,b)$ and apply (3.24), (3.25) for $\sigma = t$, $\tau \in J \cap (t,b)$ thus obtaining

$$\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)-\varepsilon}^{\xi(t)} u(x,t)dx \le v(t) \le \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)}^{\xi(t)+\varepsilon} u(x,t)dx. \tag{3.28}$$

Similarly, for fixed $t \in (a,b]$, the application of (3.24), (3.25) with $\sigma \in J \cap (a,t)$, $\tau = t$ yields

$$\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)-\varepsilon}^{\xi(t)} u(x,t)dx \ge v(t) \ge \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\xi(t)}^{\xi(t)+\varepsilon} u(x,t)dx. \tag{3.29}$$

In particular, (3.28) for $t = \alpha$ yields (3.18) and (3.29) for $t = b$ yields (3.19). Finally, for $t \in (a,b)$ both (3.28) and (3.29) apply and this leads to (3.17). The proof is complete.

Since the initial-value problem for (3.16) has a unique solution, we deduce with the help of (3.17),

Corollary 3.1. Two genuine characteristics may intersect only at their end points.

An important consequence of Corollary 3.1 is the following

Theorem 3.4. There is a unique forward characteristic through any point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$.

Proof. Suppose there were two forward characteristics $\eta(\cdot)$ and $\xi(\cdot)$ through $(\bar{x},\bar{t})$ with $\eta(\tau) < \xi(\tau)$ for some $\tau > \bar{t}$. Consider the maximal backward characteristic $\zeta_+(t;\eta(\tau),\tau)$ through $(\eta(\tau),\tau)$ and the minimal backward characteristic $\zeta_-(t;\xi(\tau),\tau)$ through $(\xi(\tau),\tau)$ which are genuine on account of Theorem 3.2. For $t \in [\bar{t},\tau]$, $\eta(t) \le \zeta_+(t;\eta(\tau),\tau)$ and $\xi(t) \ge \zeta_-(t;\xi(\tau),\tau)$. Moreover, $\eta(\bar{t}) = \xi(\bar{t}) = \bar{x}$. Therefore, $\zeta_+(t;\eta(\tau),\tau)$ and $\zeta_-(t;\xi(\tau),\tau)$ must

intersect for some $t \in [\bar{t},\tau)$ which is a contradiction to Corollary 3.1. The proof is complete.

It should be noted, however, that more than one forward characteristic may emanate from points of the axis $t = 0$.

In the next section we will employ the properties of characteristics obtained here to get information on the structure of solutions. In turn this will yield additional information on the structure of characteristics.

4. Structure of Solutions

In this section we consider an admissible solution $u(x,t)$ of (1.1) and study the geometric structure of the set of points of continuity and discontinuity.

Theorem 4.1. After, possibly, a modification on a set of measure zero, not destroying the continuity of $t \mapsto u(\cdot,t)$ required by Definition 2.1, $u(x,t)$ acquires the following properties. For each fixed $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ the one-sided limits $u(\bar{x}\pm,\bar{t})$ exist and

$$u(\bar{x}-,\bar{t}) \geq u(\bar{x}+,\bar{t}). \tag{4.1}$$

Furthermore, the minimal and maximal backward characteristics $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$ through $(\bar{x},\bar{t})$ are determined by solving (3.16) with initial conditions $(\xi(\bar{t}),v(\bar{t})) = (\bar{x},u(\bar{x}-,\bar{t}))$ and $(\xi(\bar{t}),v(\bar{t})) = (\bar{x},u(\bar{x}+,\bar{t}))$, respectively. In particular, $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$ coincide if and only if $u(\bar{x}-,\bar{t}) = u(\bar{x}+,\bar{t})$.

Proof. We fix $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$. Assume first that $\bar{x}$ is a point of approximate continuity of the function $u(\cdot,\bar{t})$ in which case

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\bar{x}-\varepsilon}^{\bar{x}} u(x,\bar{t})dx = u(\bar{x},\bar{t}) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\bar{x}}^{\bar{x}+\varepsilon} u(x,\bar{t})dx. \tag{4.2}$$

Then by Theorem 3.3 (see, in particular, (3.19)) both $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$ can be determined by solving (3.16) with initial conditions $(\xi(\bar{t}),v(\bar{t})) = (\bar{x},u(\bar{x},\bar{t}))$ and, therefore, coincide.

Suppose now $(\bar{x},\bar{t})$ arbitrary. Consider any strictly increasing sequence $\{x_n\}$, $x_n \to \bar{x}-$, such that x_n is a point of approximate continuity of $u(\cdot,\bar{t})$, $n = 1,2,\ldots$. By Corollary 3.1,

$$\zeta_\pm(t;x_1,\bar{t}) < \zeta_\pm(t;x_2,\bar{t}) < \cdots < \zeta_-(t;\bar{x},\bar{t}), \quad t \in (0,\bar{t}]. \tag{4.3}$$

The theory of contingent equations of the type (3.1) [13] asserts that $\{\zeta_\pm(t;x_n,\bar{t})\}$ must converge, uniformly in t on $[0,\bar{t}]$, to a solution of (3.1) through $(\bar{x},\bar{t})$. Since $\zeta_-(t;\bar{x},\bar{t})$ is the minimal characteristic through $(\bar{x},\bar{t})$ and by virtue of (4.3) we conclude that $\{\zeta_\pm(t;x_n,\bar{t})\}$ must converge to $\zeta_-(t;\bar{x},\bar{t})$. Since $u(x,t)$ is of class $\mathscr{K}$ (Definition 2.1),

$$u(\zeta_\pm(t;x_n,\bar{t})\pm,t) \to u(\zeta_-(t;\bar{x},\bar{t})\pm,t), \text{ a.e. on } [0,\bar{t}]. \tag{4.4}$$

On the other hand, by Theorem 3.3, $u(\zeta_\pm(t;x_n,\bar{t})\pm,t)$ coincides for almost all $t \in [0,\bar{t}]$ with a continuously differentiable function $v_n(t)$ so that $(\zeta_\pm(t;x_n,\bar{t}),v_n(t))$ is the solution of

(3.16) with initial conditions $(x_n, u(x_n,\bar{t}))$ at $t = \bar{t}$. Similarly, $u(\zeta_{\pm}(t;\bar{x},\bar{t})\pm,t)$ coincides for almost all $t \in [0,\bar{t}]$ with a continuously differentiable function $v(t)$ so that $(\zeta_-(t;\bar{x},\bar{t}),v(t))$ is a solution of (3.16). By $(3.16)_2$ $\{v_n(t)\}$ is uniformly equicontinuous on $[0,\bar{t}]$ so that (4.4) yields

$$v_n(t) \to v(t), \text{ uniformly on } [0,\bar{t}]. \tag{4.5}$$

In particular,

$$u(x_n,\bar{t}) = v_n(\bar{t}) \to v(\bar{t}). \tag{4.6}$$

If we thus modify appropriately $u(\cdot,\bar{t})$ on the set (of zero one-dimensional Lebesgue measure) of x that are not points of approximate continuity of $u(\cdot,t)$, we can guarantee that $u(\bar{x}-,\bar{t})$ exists and equals $v(\bar{t})$. Similarly, one shows that $u(\bar{x}+,\bar{t})$ exists and that $\zeta_+(t;\bar{x},\bar{t})$ can be determined by solving (3.16) with initial conditions $(\bar{x},u(\bar{x}+,\bar{t}))$ at $t = \bar{t}$.

Since $\zeta_-(t;\bar{x},\bar{t}) \leq \zeta_+(t;\bar{x},\bar{t})$, $t \in [0,\bar{t}]$, we deduce by $(3.16)_1$, $f_u(u(\bar{x}-,\bar{t}),\bar{x},\bar{t}) = \dot{\zeta}_-(\bar{t};\bar{x},\bar{t}) \geq \dot{\zeta}_+(\bar{t};\bar{x},\bar{t}) = f_u(u(\bar{x}+,\bar{t}),\bar{x},\bar{t})$ which yields (4.1). The proof is complete.

<u>Corollary 4.1</u>. For any fixed $t \in (0,\infty)$, the set of points of discontinuity of $u(\cdot,t)$ is at most countable.

We note that by virtue of Theorem 4.1 any genuine characteristic can be extended backwards, as a genuine characteristic, up to $t = 0$, i.e., all genuine characteristics emanate from the

axis $t = 0$.

We now turn our attention to characteristics that are not genuine.

Theorem 4.2. Let $\eta(\cdot)$ be a characteristic on $[\bar{t},\infty)$, $\bar{t} > 0$. If $u(\eta(\bar{t})-,\bar{t}) > u(\eta(\bar{t})+,\bar{t})$, then $u(\eta(t)-,t) > u(\eta(t)+,t)$, for all $t \geq \bar{t}$, and $u(\eta(t)-,t) - u(\eta(t)+,t)$ is bounded away from zero, uniformly on bounded subsets of $[\bar{t},\infty)$.

Proof. If for some bounded sequence $\{t_n\}$ in $[\bar{t},\infty)$, $u(\eta(t_n)-,t_n) - u(\eta(t_n)+,t_n) \to 0$, then, by Theorem 4.1 and continuous dependence of solutions of (3.16) on initial data, we would have $\zeta_-(t;\eta(t_n),t_n) - \zeta_+(t;\eta(t_n),t_n) \to 0$ for every $t \in (0,\bar{t}]$. On the other hand, on account of Corollary 3.1, $\zeta_-(t;\eta(t_n),t_n) \leq \zeta_-(t;\eta(\bar{t}),\bar{t}) < \zeta_+(t;\eta(\bar{t}),\bar{t}) \leq \zeta_+(t;\eta(t_n),t_n)$, $t \in (0,\bar{t}]$, and this induces a contradiction that proves the theorem.

Definition 4.1. A characteristic $\eta(\cdot)$ on $[\bar{t},\infty)$ is called a shock if $u(\eta(t)-,t) > u(\eta(t)+,t)$ for all $t \in (\bar{t},\infty)$.

Definition 4.2. A point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ is called a shock generation point if the (unique) forward characteristic through $(\bar{x},\bar{t})$ is a shock while every backward characteristic through $(\bar{x},\bar{t})$ is genuine on $[0,\bar{t}]$.

Definition 4.3. A point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ is called the center of a centered compression wave if there are two genuine

backward characteristics $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ through $(\bar{x},\bar{t})$ and every backward characteristic through $(\bar{x},\bar{t})$ contained in the funnel confined between $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ (*) is genuine on $[0,\bar{t}]$.

It is clear that if $(\bar{x},\bar{t})$ is a shock generation point, then either $u(\bar{x}-,\bar{t}) = u(\bar{x}+,\bar{t})$, so that $\zeta_-(t;\bar{x},\bar{t}) \equiv \zeta_+(t;\bar{x},\bar{t})$ and there is just one backward characteristic through $(\bar{x},\bar{t})$, or $u(\bar{x}-,\bar{t}) > u(\bar{x}+,\bar{t})$ in which case $(\bar{x},\bar{t})$ is the center of a centered compression wave confined between $\zeta_-(t;\bar{x},\bar{t})$ and $\zeta_+(t;\bar{x},\bar{t})$.

Our next project is to study the structure of shocks.

<u>Lemma 4.1</u>. Let $\eta(\cdot)$ be a shock defined on $[\bar{t},\infty)$, $\bar{t} > 0$. Consider the functions $w_-(t) \stackrel{\text{def}}{=} u(\eta(t)-,t)$, $w_+(t) \stackrel{\text{def}}{=} u(\eta(t)+,t)$. Then $w_\pm(\cdot)$ are continuous from the right on $[\bar{t},\infty)$. Furthermore, limits $w_\pm(t-)$ from the left exist for all $t \in (\bar{t},\infty)$ and $w_-(t-) \leq w_-(t)$, $w_+(t-) \geq w_+(t)$.

<u>Proof</u>. By Corollary 3.1, $\sigma > \tau \geq \bar{t}$ implies $\zeta_-(s;\eta(\sigma),\sigma) < \zeta_-(s;\eta(\tau),\tau)$, $s \in (0,\tau]$. In conjunction with Theorem 4.1 this shows that for fixed $t \geq \bar{t}$

$$\begin{aligned} &\zeta_-(s;\eta(\tau),\tau) \to \zeta_-(s;\eta(t),t), \quad s \in [0,t], \text{ as } \tau \to t+ \\ &u(\zeta_-(s;\eta(\tau),\tau)-,s) \to u(\zeta_-(s;\eta(t),t)-,s), \quad s \in (0,t], \text{ as } \tau \to t+. \end{aligned} \tag{4.7}$$

(*) By the theory of contingent equations of type (3.1) [13], the funnel confined between $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ will be completely filled by backward characteristics through $(\bar{x},\bar{t})$.

Similarly, for $t > \bar{t}$,

$$\begin{aligned} &\zeta_-(s;\eta(\tau),\tau) \to \zeta_0(s),\ s \in [0,t),\ \text{as}\ \ \tau \to t- \\ &u(\zeta_-(s;\eta(\tau),\tau)-,s) \to u(\zeta_0(s)+,s),\ s \in (0,t),\ \text{as}\ \ \tau \to t-, \end{aligned} \tag{4.8}$$

where $\zeta_0(\cdot)$ is a backward characteristic through $(\eta(t),t)$. On account of Theorems 3.3 and 4.1, $(\zeta_-(s;\eta(\tau),\tau),\ u(\zeta_-(s;\eta(\tau),\tau)-,s))$ is a solution of (3.16) on $(0,t)$. Thus, $(\zeta_0(s),u(\zeta_0(s)+,s))$ is also a solution of (3.16) on $(0,t)$. By virtue of Theorem 3.1 this implies that $\zeta_0(\cdot)$ is genuine on $(0,t)$. We also conclude that the convergence in (4.7) and (4.8) is uniform. In particular, the limits $w_-(t\pm)$ exist and $w_-(t+) = w_-(t)$. Finally, $f_u(w_-(t),\eta(t),t) = \dot{\zeta}_-(t;\eta(t),t) \geq \dot{\zeta}_0(t) = f_u(w_-(t-),\eta(t),t)$ so that $w_-(t) \geq w_-(t-)$. The analogous properties of $w_+(\cdot)$ are established by the same procedure. The proof is complete.

In particular, the set of points of discontinuity of $w_\pm(\cdot)$ is at most countable. If t is a point of continuity of $w_-(\cdot)$ the backward characteristic $\zeta_0(s)$ constructed in the proof of Lemma 4.1 coincides with $\zeta_-(s;\eta(t),t)$. If, however, $w_-(t-) < w_-(t)$, then $\zeta_-(s;\eta(t),t) < \zeta_0(s)$, $s \in (0,t)$. In this case the funnel confined between $\zeta_-(s;\eta(t),t)$ and $\zeta_0(s)$ is spanned by backward characteristics through $(\eta(t),t)$. If all these characteristics are genuine, $(\eta(t),t)$ is the center of a centered compression wave; otherwise $(\eta(t),t)$ is a point of shocks interaction. Analogous arguments apply to $w_+(\cdot)$. These considerations together with Theorem 3.1 and Lemma 4.1 yield

Theorem 4.3. Let $\eta(\cdot)$ *be a shock on* $[\bar{t},\infty)$, $\bar{t} > 0$. Then the derivative of $\eta(\cdot)$ from the right exists at every $t \in [\bar{t},\infty)$ and is given by

$$D^+\eta(t) = \frac{f(u(\eta(t)+,t),\eta(t),t) - f(u(\eta(t)-,t),\eta(t),t)}{u(\eta(t)+,t) - u(\eta(t)-,t)} . \quad (4.9)$$

Furthermore, $\dot{\eta}(t)$ exists and is continuous except on the (at most countable) set of points of interaction of $\eta(\cdot)$ with another shock and/or a centered compression wave. At every point of interaction the strength $u(\eta(t)-,t) - u(\eta(t)+,t)$ of the shock increases.

In particular, any shock may interact with an at most countable set of other shocks. This observation combined with Corollary 4.1 yields

Corollary 4.2. The set of shocks is at most countable.

We now investigate the smoothness of $u(x,t)$, jointly in the (x,t) variables.

Theorem 4.4. Let $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ and let $\eta(\cdot)$ be the (unique) forward characteristic through $(\bar{x},\bar{t})$. Then, $(\bar{x},\bar{t})$ is a point of continuity of $u(\bar{x},\bar{t})$ relative to the set $\mathscr{S}_- = \{(x,t) \mid t \geq \bar{t},\ x < \eta(t)$ and $t < \bar{t},\ x \leq \zeta_-(t;\bar{x},\bar{t})\}$, the limit being $u(\bar{x}-,\bar{t})$, and also relative to the set $\mathscr{S}_+ = \{(x,t) \mid t \geq \bar{t},\ x > \eta(t)$ and $t < \bar{t},\ x \geq \zeta_+(t;\bar{x},\bar{t})\}$ with limit $u(\bar{x}+,\bar{t})$.

Proof. Every sequence in $\mathcal{S}_-$ converging to $(\bar{x},\bar{t})$ must contain a subsequence $\{(x_n,t_n)\}$ such that $\{\zeta_-(t;x_n,t_n)\}$ converges to a backward characteristic through $(\bar{x},\bar{t})$. Since $\zeta_-(t;x,s) \leq \zeta_-(t;\bar{x},\bar{t})$, for any $(x,s) \in \mathcal{S}_-$, it follows that $\zeta_-(t;x_n,t_n) \to \zeta_-(t;\bar{x},\bar{t})$, $t \in (0,\bar{t})$. Consequently, $u(\zeta_-(t;x_n,t_n)\pm,t) \to u(\zeta_-(t;\bar{x},\bar{t})-,t)$, $t \in (0,\bar{t})$. As in the proof of Lemma 4.1, we show with the help of Theorems 3.3 and 4.1 that the convergence, above, is uniform. In particular, $u(x_n\pm,t_n) \to u(\bar{x}-,\bar{t})$. Similarly one shows that $(\bar{x},\bar{t})$ is a point of continuity of $u(x,t)$ relative to the set $\mathcal{S}_+$ and the limit is $u(\bar{x}+,\bar{t})$. The proof is complete.

Corollary 4.3. The solution $u(x,t)$ is continuous at the point $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ if and only if $u(\bar{x}-,\bar{t}) = u(\bar{x}+,\bar{t})$.

Theorem 4.5. Let $\eta(\cdot)$ be a shock and let $\bar{t}$ be a point of continuity of the functions $w_\pm(t) = u(\eta(t)\pm,t)$. Then $(\eta(\bar{t}),\bar{t})$ is a point of continuity of $u(x,t)$ relative to the set $\mathcal{S}_- = \{(x,t) \mid x < \eta(t)\}$, the limit being $u(\eta(\bar{t})-,\bar{t})$, and also relative to the set $\mathcal{S}_+ = \{(x,t) \mid x > \eta(t)\}$ with limit $u(\eta(\bar{t})+,\bar{t})$.

Proof. Any sequence in $\mathcal{S}_-$ converging to $(\eta(\bar{t}),\bar{t})$ must contain a subsequence $\{(x_n,t_n)\}$ such that $\{\zeta_-(t;x_n,t_n)\}$ converges to a backward characteristic through $(\eta(\bar{t}),\bar{t})$. By Corollary 3.1, $\zeta_-(t;x_n,t_n) < \zeta_-(t;\eta(t_n),t_n)$, $t \in (0,t_n]$. Moreover, (see the discussion following the proof of Lemma 4.1) since $\bar{t}$ is a point of continuity of $w_-(\cdot)$, $\zeta_-(t;\eta(t_n),t_n) \to \zeta_-(\bar{t};\eta(\bar{t}),t)$, $t \in (0,\bar{t})$. Therefore, $\zeta_-(t;x_n,t_n) \to \zeta_-(t;\eta(\bar{t}),\bar{t})$, $t \in (0,\bar{t})$. Theorem 4.1

yields $u(\zeta_-(t;x_n,t_n)-,t) \to u(\zeta_-(t;\eta(\bar{t}),\bar{t})-,t)$, $t \in (0,\bar{t})$, and the convergence is uniform. In particular, $u(x_n-,t_n) \to u(\eta(\bar{t})-,\bar{t})$. Similarly one shows that $(\eta(\bar{t}),\bar{t})$ is also a point of continuity of $u(x,t)$ relative to the set $\mathcal{S}_+$ and the limit is $u(\eta(\bar{t})+,\bar{t})$. The proof is complete.

From the above results the following picture of the structure of solutions emerges: There is an at most countable set of shock With the exception of interaction points, limits of the solution exist on both sides of a shock and the Rankine-Hugoniot condition is satisfied. On the complement of the shock set the solution is continuous.

In the next section we will study smoothness of solutions when the initial dates are smooth.

5. Smoothness of Solutions

In this section we assume that $f(u,x,t)$ and $g(u,x,t)$ are, respectively, C^{k+1} and C^k smooth, $3 \leq k \leq \infty$, and $f(\cdot,x,t)$ is locally uniformly convex(*), i.e., for all (u,x,t),

$$f_{uu}(u,x,t) > 0. \tag{5.1}$$

We study the smoothness of admissible solutions $u(x,t)$ of (1.1) with C^k smooth initial data

(*) See Remark 5.1, below.

$$u(x,0) = w(x), \quad x \in (-\infty,\infty). \tag{5.2}$$

For y and w in $(-\infty,\infty)$ we let $(\xi(t,y,w),v(t,y,w))$ denote the solution of (3.16) with initial conditions $\xi(0,y,w) = y$, $v(0,y,w) = w$. We note that $\xi(t,y,w)$ and $v(t,y,w)$ are C^k smooth functions on their domain of definition (t in the maximal interval of existence of $(\xi(\cdot,y,w),v(\cdot,y,w))$).

For $(x,t) \in (-\infty,\infty) \times (0,\infty)$ we consider the interceptors $\zeta_-(0;x,t)$ and $\zeta_+(0;x,t)$ of the backward minimal and maximal characteristics through (x,t). On account of Theorem 4.1,

$$x = \xi(t,\zeta_\pm(0;x,t),w(\zeta_\pm(0;x,t))), \tag{5.3}$$

$$u(x\pm,t) = v(t,\zeta_\pm(0;x,t),w(\zeta_\pm(0;x,t))). \tag{5.4}$$

By the arguments employed repeatedly in Section 4, it is clear that, for fixed $\bar{t} \in (0,\infty)$, $\zeta_-(0;x,\bar{t})$ and $\zeta_+(0;x,\bar{t})$ are strictly increasing functions of x, continuous from the left and right, respectively. In particular, for any $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$,

$$\frac{\partial\xi(\bar{t},y,w(y))}{\partial y} \geq 0, \quad y = \zeta_\pm(0;\bar{x},\bar{t}). \tag{5.5}$$

It is also clear that every point $(\bar{x},\bar{t})$ of continuity of $u(x,t)$ is also a point of continuity of $\zeta_\pm(0;x,t)$. (Of course, at a point of continuity $\zeta_-(0;\bar{x},\bar{t}) = \zeta_+(0;\bar{x},\bar{t})$). From this observation, (5.3), (5.4) and the implicit function theorem we deduce

Lemma 5.1. If $(\bar{x},\bar{t}) \in (-\infty,\infty) \times (0,\infty)$ is a point of continuity of $u(x,t)$ and

$$\frac{\partial\xi(\bar{t},y,w(y))}{\partial y} > 0, \quad y = \zeta_{\pm}(0;\bar{x},\bar{t}), \tag{5.6}$$

then $u(x,t)$ is C^k smooth on a neighborhood of $(\bar{x},\bar{t})$.

From (5.5) and Lemma 5.1 we obtain the following corollary.

Lemma 5.2. If $(\bar{x},\bar{t})$ is a point of continuity of $u(x,t)$ as well as a shock generation point, then

$$\frac{\partial\xi(\bar{t},y,w(y))}{\partial y} = 0, \quad y = \zeta_{\pm}(0;\bar{x},\bar{t}). \tag{5.7}$$

We now consider a point $(\bar{x},\bar{t})$ of discontinuity of $u(x,t)$. If $(\bar{x},\bar{t})$ lies on a shock $\eta(\cdot)$ and $\bar{t}$ is a point of continuity of $u(\eta(t)\pm,t)$, then, using Theorem 4.5, (5.3), (5.4) and the implicit function theorem we conclude that, if (5.6) is satisfied, then there are C^k smooth functions $u_-(x,t)$ and $u_+(x,t)$, defined on some neighborhood $\mathcal{N}$ of $(\bar{x},\bar{t})$, such that $u(x,t) = u_-(x,t)$ on $\{(x,t) \in \mathcal{N} \mid x < \eta(t)\}$ and $u(\bar{x},\bar{t}) = u_+(x,t)$ on $\{(x,t) \in \mathcal{N} \mid x > \eta(t)\}$. In conjunction with Theorem 4.3 this yields

Lemma 5.3. If $(\bar{x},\bar{t})$ is a point on a shock $\eta(\cdot)$, (5.6) is satisfied and $\bar{t}$ is a point of continuity of $u(\eta(t)\pm,t)$, then $\eta(\cdot)$ is C^{k+1} smooth on a neighborhood of $\bar{t}$ and $u(x,t)$ is C^k smooth on either side of $\eta(\cdot)$, on some neighborhood of $(\bar{x},\bar{t})$.

Finally, we consider the case where $(\bar{x},\bar{t})$ is the center of a centered compression wave. By Definition 4.3,

Lemma 5.4. If $(\bar{x},\bar{t})$ is the center of a centered compression wave confined between the genuine characteristics $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, then

$$\xi(\bar{t},y,w(y)) = \bar{x} \text{ for all } y \in [\zeta_1(0),\zeta_2(0)]. \tag{5.8}$$

In particular,

$$\frac{\partial\xi(\bar{t},y,w(y))}{\partial y} = 0 \quad \text{for all} \quad y \in [\zeta_1(0),\zeta_2(0)]. \tag{5.9}$$

Our next project is to study the structure of the set of points for which (5.7) is satisfied. To this end we introduce the function

$$F(t,y,w,\omega) \overset{\text{def}}{=} \xi_y(t,y,w) + \omega\xi_w(t,y,w), \tag{5.10}$$

defined for x,w,ω in $(-\infty,\infty)$ and t in the maximal interval of existence of $(\xi(\cdot,y,w),v(\cdot,y,w))$, and we observe that

$$\frac{\partial\xi(t,y,w(y))}{\partial y} = F(t,y,w(y),w'(y)). \tag{5.11}$$

Lemma 5.5. If at a point $(\bar{t},\bar{y},\bar{w},\bar{\omega})$,

$$F(\bar{t},\bar{y},\bar{w},\bar{\omega}) = 0, \tag{5.12}$$

then

$$F_t(\bar{t},\bar{y},\bar{w},\bar{\omega}) \neq 0. \tag{5.13}$$

Furthermore,

$$\xi_w(\bar{t},\bar{y},\bar{w}) \neq 0. \tag{5.14}$$

Proof. The matrix-valued function

$$Y(t,y,w) = \begin{bmatrix} \xi_y(t,y,w) & \xi_w(t,y,w) \\ v_y(t,y,w) & v_w(t,y,w) \end{bmatrix} \tag{5.15}$$

is the principal fundamental solution at $t = 0$ of the linear variational system of (3.16), viz.,

$$\begin{aligned} Y_t(t,y,w) &= A(t,y,w)Y(t,y,w) \\ Y(0,y,w) &= \text{identity matrix} \end{aligned} \tag{5.16}$$

where

$$A(t,y,w) = \begin{bmatrix} f_{ux} & f_{uu} \\ -f_{xx}-g_x & -f_{ux}-g_u \end{bmatrix}, \tag{5.17}$$

evaluated at $u = v(t,y,w)$, $x = \xi(t,y,w)$ and t. Therefore, $Y(t,y,w)$ is nonsingular for all t in the domain of definition. By virtue of (5.10) and (5.12) this implies that (5.14) is satisfied and also

$$v_y(\bar{t},\bar{y},\bar{w}) + \bar{\omega} v_w(\bar{t},\bar{y},\bar{w}) \neq 0. \tag{5.18}$$

On the other hand, from (5.10) and (5.15), (5.16), (5.17), we obtain

$$\begin{aligned} F_t(t,y,w,\omega) &= \xi_{yt}(t,y,w) + \omega\xi_{wt}(t,y,w) \\ &= f_{ux}(v(t,y,w),\xi(t,y,w),t)[\xi_y(t,y,w) + \omega\xi_w(t,y,w)] \\ &+ f_{uu}(v(t,y,w),\xi(t,y,w),t)[v_y(t,y,w) + \omega v_w(t,y,w)]. \end{aligned} \tag{5.19}$$

Using (5.12), (5.10), (5.18) and (5.1) we arrive at (5.13). The proof is complete.

Remark 5.1. The sole purpose of our assumption (5.1) is the derivation of (5.13) from (5.19). It is clear, however, that this assumption may be replaced by the weaker one: $\xi_w(\bar{t},\bar{y},\bar{w}) \neq 0$ implies $f_{uu}(v(\bar{t},\bar{y},\bar{w}),\xi(\bar{t},\bar{y},\bar{w}),\bar{t}) \neq 0$. This is satisfied, for example, for (1.2) even if $f(u)$ is not uniformly convex. Indeed, in this case integration of (3.16) yields $\xi(t,y,w) = y + tf_u(w)$, $v(t,y,w) = w$ so that $\xi_w(t,y,w) = tf_{uu}(w)$. For a detailed study of this special case we refer to [14].

On account of (5.11) and Lemma 5.5 we conclude that, for fixed y, $\partial\xi(t,y,w(y))/\partial y$, as a function of t, must change sign across its zeros. In combination with (5.3) and (5.5) this implies that if (5.5) is satisfied as an equality at a point $(\bar{x},\bar{t})$ then the (unique) forward characteristic through $(\bar{x},\bar{t})$ is necessarily

a shock, i.e., $(\bar{x},\bar{t})$ is a point of the shock set. Therefore, (5.5) must be satisfied as a strict inequality on the complement of the shcok set in which case Lemma 5.1 yields

<u>Theorem 5.1</u>. The complement of the shock set is open and $u(x,t)$ is C^k smooth on this set.

In order to study the structure of the set of shock generation points and/or centers of centered compression waves, we let U denote the set of (y,w,ω) with the property that there is positive $T(y,w,\omega)$ in the maximal interval of existence of $(\xi(\cdot,y,w),v(\cdot,y,w))$ such that

$$\begin{aligned} &F(t,y,w,\omega) > 0 \qquad 0 \leq t < T(y,w,\omega) \\ &F(t,y,w,\omega) = 0, \qquad t = T(y,w,\omega). \end{aligned} \tag{5.20}$$

(Note that (5.10), (5.15) and (5.16) imply $F(0,y,w,\omega) = 1$ for all (y,w,ω)). From Lemma 5.5 and the implicit function theorem we conclude that U is open and $T(y,w,\omega)$ is C^{k-1} smooth on U.

<u>Lemma 5.6</u>. If $(\bar{x},\bar{t})$ is a point of continuity of $u(x,t)$ as well as a shock generation point and $\bar{y} = \zeta_{\pm}(0;\bar{x},\bar{t})$, then $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$ and $\bar{t} = T(\bar{y},w(\bar{y}),w'(\bar{y}))$. Furthermore, $\bar{y}$ is a critical point of the function $T(y,w(y),w'(y))$.

<u>Proof</u>. From Lemma 5.2, (5.11), and the definition of U and T we conclude that $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$, $\bar{y} = \zeta_{\pm}(0;\bar{x},\bar{t})$, and

$T(\bar{y},w(\bar{y}),w'(\bar{y})) = \bar{t}$. We now let $\eta(\cdot)$ denote the shock generated at $(\bar{x},\bar{t})$. We consider a strictly decreasing sequence $\{t_n\}$, $t_n \to \bar{t}+$, and we set $y_n^- = \zeta_-(0;\eta(t_n),t_n)$, $y_n^+ = \zeta_+(0;\eta(t_n),t_n)$. By virtue of Lemma 4.1, $y_n^- \to \bar{y}-$, $y_n^+ \to \bar{y}+$. Therefore, for n sufficiently large, $(y_n^{\pm},w(y_n^{\pm}),w'(y_n^{\pm})) \in U$. Furthermore, by (5.5), $T(y_n,w(y_n),w'(y_n)) \geq t_n > \bar{t}$. Hence $\bar{y}$ is a critical point of $T(y,w(y),w'(y))$. The proof is complete.

Similarly, on account of Lemma 5.4,

<u>Lemma 5.7</u>. If $(\bar{x},\bar{t})$ is the center of a centered compression wave confined between the genuine characteristics $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, then, for any $\bar{y} \in [\zeta_1(0),\zeta_2(0)]$, $(y,w(y),w'(y)) \in U$ and $\bar{t} = T(\bar{y},w(\bar{y}),w'(\bar{y}))$. In particular, every point of $[\zeta_1(0),\zeta_2(0)]$ is a critical point of $T(y,w(y),w'(y))$.

We will say that the solution $u(x,t)$ is <u>piecewise smooth</u> if every bounded subset of $(-\infty,\infty) \times [0,\infty)$ intersects an at most finite number of shocks, every shock is a piecewise continuously differentiable curve, and $u(x,t)$ is C^k smooth on the complement of the shock set.

The following proposition exhibits a property of the function $T(y,w(y),w'(y))$ when the solution generated by $w(\cdot)$ is not piecewise smooth.

<u>Lemma 5.8</u>. Suppose that $u(x,t)$ is not piecewise smooth. Then the function $T(y,w(y),w'(y))$ has a degenerate critical point. In fact, there is $\bar{y}$ such that $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$ and

$$\frac{d^m}{dy^m} T(y,w(y),w'(y)) = 0, \quad y = \bar{y},\ m = 1,\ldots,k-1. \quad (5.21)$$

Proof. Since $u(x,t)$ is not piecewise smooth, there are numbers $\bar{T} > 0$ and $-\infty < a < b < \infty$ so that the set $S = \{(x,t) \mid 0 \leq t < \bar{T}, \zeta_-(t;a,\bar{T}) < x < \zeta_+(t;b,\bar{T})\}$ has the following property: S is intersected by an infinite set of shocks and/or at least one of the shocks intersecting S is not piecewise C^1 smooth in S. We note that the number of shocks intersecting S equals the number of shock generation points contained in S. Furthermore, by Theorem 4.3, differentiability of a shock may only fail at shock interaction points and/or centers of centered compression waves. Therefore, using Lemmas 5.6 and 5.7, we conclude that $T(y,w(y),w'(y))$ must have an infinite number of critical points inside the interval $(\zeta_-(0;a,\bar{T}),\zeta_+(0;b,\bar{T}))$, corresponding to shock generation points and/or centers of centered compression waves.

We select a convergent sequence $\{y_n\}$ from the above set of critical points with limit, say, $\bar{y}$ and such that $\{T(y_n,w(y_n),w'(y_n))\}$ is also convergent to, say, $\bar{t} \in [0,\bar{T}]$. Since S is bounded and $u(x,t)$ is locally bounded on $(-\infty,\infty) \times [0,\infty)$, we deduce that for n sufficiently large the sequence $\{(\xi(t,y_n,w(y_n)),v(t,y_n,w(y_n)))\}$ is defined and is uniformly bounded on an interval $[0,\bar{t}+\varepsilon]$, with ε positive small. Therefore, $\bar{t}$ lies in the maximal interval of existence of $(\xi(\cdot,\bar{y},w(\bar{y})),v(\cdot,\bar{y},w(\bar{y})))$ and

$$\begin{aligned} F(t,\bar{y},w(\bar{y}),w'(\bar{y})) &= \lim_{n\to\infty} F(t,y_n,w(y_n),w'(y_n)) \geq 0, \quad t \in [0,\bar{t}) \\ F(\bar{t},\bar{y},w(\bar{y}),w'(\bar{y})) &= \lim_{n\to\infty} F(T(y_n,w(y_n),w'(y_n)),y_n,w(y_n),w'(y_n)) = 0. \end{aligned} \quad (5.22)$$

By virtue of Lemma 5.5, $(5.22)_1$ must be satisfied as a strict inequality on $[0,\bar{t})$. Hence, by the definition of U and T, $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$ and $T(\bar{y},w(\bar{y}),w'(\bar{y})) = \bar{t}$.

Since U is open, $T(y,w(y),w'(y))$ is defined and is C^{k-1} smooth on an interval $(\bar{y}-\delta,\bar{y}+\delta)$. Noting that $y_n \in (\bar{y}-\delta,\bar{y}+\delta)$ for n large, we conclude that derivatives of $T(y,w(y),w'(y))$ of order $1,\ldots,k-1$ must vanish at $\bar{y}$ and this proves the Lemma.

We now have the preparation to show that solutions are generically piecewise smooth. Several variants of this result can be established, depending on the choice of topology. We will work here in the space C^k of functions that are bounded on $(-\infty,\infty)$ together with their derivatives of order $1,\ldots,k$, equipped with the topology induced by uniform convergence on $(-\infty,\infty)$ of derivatives of order $0,1,\ldots,k$.

Theorem 5.2. Admissible solutions of (1.1) generated by initial data in C^k, $3 \le k \le \infty$, are generically piecewise smooth.

Proof. In view of Lemma 5.8 we have to show that the set of $w(\cdot)$ in C^k for which $T(y,w(y),w'(y))$ has degenerate critical points is of the first category in C^k. To this end it suffices to prove that for any fixed interval $[a,b]$ and fixed positive numbers $\bar{T},\bar{v},\bar{\xi}$ the set of $w(\cdot)$ in C^k for which there is $\bar{y} \in [a,b]$, with $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$, satisfying

$$T(\bar{y},w(\bar{y}),w'(\bar{y})) \le \bar{T} \qquad (5.23)$$

$$\left.\begin{aligned} |\xi(t,\bar{y},w(\bar{y}))| &\le \bar{\xi} \\ |v(t,\bar{y},w(\bar{y}))| &\le \bar{v} \end{aligned}\right\} \quad t \in [0,T(\bar{y},w(\bar{y}),w'(\bar{y}))] \tag{5.24}$$

$$\frac{d^m}{dy^m} T(y,w(y),w'(y)) = 0, \; y = \bar{y}, \; m = 1,2, \tag{5.25}$$

is closed and nowhere dense in C^k.

We first show that the set is closed. Let $\{w_n(\cdot)\}$ be a sequence in C^k so that for each n there is $y_n \in [a,b]$, with $(y_n,w_n(y_n),w_n'(y_n)) \in U$, satisfying

$$T(y_n,w_n(y_n),w_n'(y_n)) \le \bar{T} \tag{5.26}$$

$$\left.\begin{aligned} |\xi(t,y_n,w_n(y_n))| &\le \bar{\xi} \\ |v(t,y_n,w_n(y_n))| &\le \bar{v} \end{aligned}\right\} \quad t \in [0,T(y_n,w_n(y_n),w_n'(y_n))] \tag{5.27}$$

$$\frac{d^m}{dy^m} T(y,w_n(y),w_n'(y)) = 0, \; y = y_n, \; m = 1,2, \tag{5.28}$$

and let $w_n(\cdot) \xrightarrow{C^k} w(\cdot)$. Without loss of generality assume $y_n \to \bar{y} \in [a,b]$, $T(y_n,w_n(y_n),w_n'(y_n)) \to \bar{t} \in [0,\bar{T}]$. In view of (5.27), $\xi(t,y_n,w_n(y_n)) \to \xi(t,\bar{y},w(\bar{y}))$, $v(t,y_n,w_n(y_n)) \to v(t,\bar{y},w(\bar{y}))$, uniformly for $t \in [0,\bar{t}]$. It follows that $\bar{t}$ is contained in the maximal interval of existence of $(\xi(t,\bar{y},w(\bar{y})),v(t,\bar{y},w(\bar{y})))$. Furthermore,

$$\begin{aligned} F(t,\bar{y},w(\bar{y}),w'(\bar{y})) &= \lim_{n\to\infty} F(t,y_n,w_n(y_n),w_n'(y_n)) \geq 0, \quad t \in [0,\bar{t}) \\ F(\bar{t},\bar{y},w(\bar{y}),w'(\bar{y})) &= \lim_{n\to\infty} F(T(y_n,w_n(y_n),w_n'(y_n)),y_n,w_n(y_n),w_n'(y_n)) = 0. \end{aligned} \tag{5.29}$$

On account of Lemma 5.5, $(5.29)_1$ must be satisfied as a strict inequality on $[0,\bar{t})$. Hence $(\bar{y},w(\bar{y}),w'(\bar{y})) \in U$ and $T(\bar{y},w(\bar{y}),w'(\bar{y})) = \bar{t}$. Obviously (5.24) is satisfied. Partial derivatives of order $\leq k - 1$ of $T(y,w,\omega)$, evaluated at $(y_n,w_n(y_n),w_n'(y_n))$, will converge to the corresponding derivatives evaluated at $(\bar{y},w(\bar{y}),w'(\bar{y}))$, because T is C^{k-1} smooth. Thus, (5.25) is obtained by taking the limit in (5.28). This completes the proof that the set is closed.

In order to prove that the same set is also nowhere dense, we fix $w(\cdot)$ in C^k and proceed to show that there are functions in C^k, arbitrarily near $w(\cdot)$, for which (5.23), (5.24), (5.25) are not jointly satisfied for any $\bar{y} \in [a,b]$.

The set of y with the properties: $(y,w(y),w'(y)) \in U$,

$$T(y,w(y),w'(y)) < \bar{T} + 1 \tag{5.30}$$

$$\left.\begin{aligned} |\xi(t,y,w(y))| &< \bar{\xi} + 1 \\ |v(t,y,w(y))| &< \bar{v} + 1 \end{aligned}\right\} \quad t \in [0,T(y,w(y),w'(y))] \tag{5.31}$$

is an open covering of the compact set of $\bar{y} \in [a,b]$ that satisfy (5.23), (5.24). Hence, there is a finite subcovering. We can thus find numbers $a \leq a_1 < b_1 < \ldots < a_n < b_n \leq b$ with the following properties: the set of $\bar{y}$ satisfying (5.23), (5.24) is

contained in $\bigcup_{i=1}^{n} [a_i,b_i]$; for each $y \in [a_i,b_i]$, $i = 1,\dots,n$,

$$T(y,w(y),w'(y)) \leq \bar{T} + 1, \tag{5.32}$$

$$\left.\begin{aligned} |\xi(t,y,w(y))| &\leq \bar{\xi} + 1 \\ |v(t,y,w(y))| &\leq \bar{v} + 1 \end{aligned}\right\} \quad t \in [0,T(y,w(y),w'(y))]; \tag{5.33}$$

finally, for $y = a_i$ and $y = b_i$ and with the possible exception of $y = a_1$, if $a_1 = a$, and/or $y = b_n$, if $b_n = b$, at least one of (5.32), $(5.33)_1$, $(5.33)_2$ holds as an equality.

With each $[a_i,b_i]$ we associate $\varepsilon_i > 0$, with $\varepsilon_i < \frac{1}{2}(a_i-b_{i-1})$, $i = 2,\dots,n$, and such that $(y,w(y),w'(y)) \in U$ for any $y \in [a_i-\varepsilon_i,a_i]$. We construct C^∞ functions $\phi_i(y)$ with the following properties: $\phi_i(\cdot)$ is near zero in C^∞; the support of $\phi_i(\cdot)$ is contained in the interval $(a_i-\varepsilon_i,\infty)$; all critical points, if any, of the function $T(y,w(y),w'(y)) + \phi_i(y)$ on the interval $[a_i,b_i]$ are nondegenerate.(*)

We now construct a function $\bar{w}(\cdot)$ in C^k as follows: We set $\bar{w}(y) = w(y)$ for $y \leq a_1 - \varepsilon_1$. On $[a_i-\varepsilon_i,b_i]$, $\bar{w}(\cdot)$ is defined as the solution of the initial-value problem

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(*) It is well-known that the set A of values of $\frac{d}{dy} T(y,w(y),w'(y))$ at points where $\frac{d^2}{dy^2} T(y,w(y),w'(y)) = 0$ has Lebesgue measure zero. Hence, if we arrange so that $\phi_i(y) = -ky$, $y \in [a_i,b_i]$, where $k \notin A$, we have $\frac{d}{dy}[T(y,w(y),w'(y)) + \phi_i(y)] \neq 0$ if $\frac{d^2}{dy^2}[T(y,w(y),w'(y)) + \phi_i(y)] = 0$, $y \in [a_i,b_i]$.

$$\xi_w(T(y,w(y),w'(y)) + \phi_i(y),y,\bar{w}(y)) \frac{d\bar{w}(y)}{dy}$$

$$+ \xi_x(T(y,w(y),w'(y)) + \phi_i(y),y,\bar{w}(y)) = 0 , \tag{5.34}$$

$$\bar{w}(a_i-\varepsilon_i) = w(a_i-\varepsilon_i) .$$

Finally, on $(b_i,a_{i+1}-\varepsilon_{i+1})$, $i = 1,\ldots,n-1$, and on (b_n,∞) we define $\bar{w}(\cdot)$ in such a way that $\bar{w}(\cdot)$ is C^k on $(-\infty,\infty)$ and is near $w(\cdot)$ in C^k.

We note that the above construction of $\bar{w}(\cdot)$ is possible for the following reason: For any $y \in [a_i-\varepsilon_i,b_i]$, $(y,w(y),w'(y)) \in U$ and by virtue of Lemma 5.5 we know that $\xi_w(T(y,w(y),w'(y)),y,w(y))$ is bounded away from zero uniformly on $[a_i-\varepsilon_i,b_i]$. Moreover, on account of (5.11) and the definition of T, $\bar{w}(\cdot) = w(\cdot)$ is the solution of (5.34) if $\phi_i(\cdot) = 0$. Therefore, for $\phi_i(\cdot)$ sufficiently small, the initial-value problem (5.34) is well-posed and its solution $\bar{w}(\cdot)$ will be near $w(\cdot)$ on $[a_i-\varepsilon_i,b_i]$ in C^k.

We also observe that for $\bar{w}(\cdot)$ sufficiently near $w(\cdot)$ the set of $y \in [a,b]$ satisfying

$$T(y,\bar{w}(y),\bar{w}'(y)) \le \bar{T}, \tag{5.35}$$

$$\left.\begin{aligned} |\xi(t,y,\bar{w}(y))| &\le \bar{\xi} \\ |v(t,y,\bar{w}(y))| &\le \bar{v} \end{aligned}\right\} \quad t \in [0,T(y,\bar{w}(y),\bar{w}'(y))] \tag{5.36}$$

is contained in $\bigcup_{i=1}^{n} [a_i,b_i]$. Finally, on account of (5.11), (5.34) and the definition of T,

$$T(y,\overline{w}(y),\overline{w}'(y)) = T(y,w(y),w'(y)) + \phi_i(y), \quad y \in [a_i-\varepsilon_i,b_i]. \tag{5.37}$$

Therefore, by the construction of $\phi_i(\cdot)$ we conclude that no $y \in [a,b]$, that satisfies (5.35), (5.36), can be a degenerate critical point of $T(y,w(y),w'(y))$. This shows that the set of $w(\cdot)$ satisfying (5.23), (5.24) and (5.25) for some $\overline{y} \in [a,b]$ is nowhere dense in C^k. The proof is complete.

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